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GELFAND–KIRILLOV DIMENSION UNDER BASE FIELD EXTENSION

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ABSTRACT

Let $F \subset K$ be a field extension, A be a K-algebra. It is proved that, in general, $\mathrm{GK}\dim_F A \geq \mathrm{GK}\dim_K A + \mathrm{tr}_F(K)$. For commutative algebras or Noetherian P.I. algebras, the equality holds. Two examples are also constructed to show that: (i) there exists an algebra A such that $\mathrm{GK}\dim_F A = \mathrm{GK}\dim_K A + \mathrm{tr}_F(K) + 1$; (ii) there exists an algebraic extension $F \subset K$ and a K-algebra A such that $\mathrm{GK}\dim_F A = \infty$, but $\mathrm{GK}\dim_K A < \infty$.

1. Introduction

The purpose of this note is to contribute to the following problem on Gelfand–Kirillov dimension: If $F \subset K$ is a field extension, A is an algebra over K, what can we say about the relation between $\mathrm{GK}\dim_F A$ and $\mathrm{GK}\dim_K A$ (A is viewed as an F-algebra naturally). First, let us recall some definitions. Let A be a finitely generated algebra over the field K and let V be a finite dimensional generating subspace of A_K, i.e., $A = K[V]$, the subalgebra of A generated by V. The Gelfand–Kirillov dimension of A is defined to be

$$\mathrm{GK}\dim_K A = \overline{\lim_{n\to\infty}} \log_n \dim_K(K + V + \cdots + V^n) = \overline{\lim_{n\to\infty}} \log_n d_V(n)$$

(conventionally, $d_V(n) = \dim_K(K + V + \cdots + V^n)$). This is equal to

$$\mathrm{Inf}\{r \in R \mid d_V(n) \leq n^r \text{ for almost all } n\}.$$

For any K-algebra A (not necessarily finitely generated), $\mathrm{GK}\dim_K A$ is defined to be $\mathrm{Sup}\{\mathrm{GK}\dim_K A' \mid \text{all finitely generated subalgebras } A' \text{ of } A\}$ ([3,5]). By definition, in fact, Gelfand–Kirillov dimension measures the rate of growth of the steps

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of V generating A. GK $\dim_K A$ is independent of the choice of the generating subspace of A_K. It is an invariant of the K-algebra A. Although in some special cases, GK $\dim_K A$ = cl.K dim A, it is easy to see that GK $\dim_K A$ depends on the base field K. GK $\dim_K A$ is not an invariant of the ring A. The general theory of Gelfand–Kirillov dimension has some similarity with the theory of Krull dimension on some occasions, but Gelfand–Kirillov dimension and Krull dimension are fundamentally different. Krull dimension is a ring-theoretic invariant. In some cases, Gelfand–Kirillov dimension has more advantage than Krull dimension. The theory of Gelfand–Kirillov dimension, developed since Borho and Kraft's paper ([3]) in 1976, has many successful applications in several important classes of algebras. Recently, people are interested in seeking the ring-theoretic properties from the information given by the Gelfand–Kirillov dimension, for example, if GK $\dim_K A$ is finite, is the length of the descending chain of prime ideals of A bounded ([2])? From this kind of problem, our problem about the behavior of Gelfand–Kirillov dimension under base field extension arises. Let $F \subset K$ be a field extension, A be a K-algebra. In this paper, we prove GK $\dim_F A \geq$ GK $\dim_K A + \mathrm{tr}_F(K)$ in general. If A is a commutative algebra or a Noetherian P.I. algebra, we have the equality GK $\dim_F A =$ GK $\dim_K A + \mathrm{tr}_F(K)$, where $\mathrm{tr}_F(K)$ is the transcendental degree of the field K over F.

We also construct two examples to show the above inequality can be strict. One is an algebra A with two generators over $K(X)$, such that GK $\dim_K A =$ GK $\dim_{K(X)} A + 2$. The other shows that there exists an algebra A over an infinite algebraic extension field K over F, such that GK $\dim_K A = 3$, but GK $\dim_F A = \infty$.

2. Proof of the results

First, we consider the case of a simple transcendental extension.

Proposition 1. *Let A be an algebra over $K(X)$ where $K(X)$ is the field of rational functions in one indeterminate X over K, then* GK $\dim_K A \geq$ GK $\dim_{K(X)} A + 1$.

Proof. We may assume that A is a finitely generated $K(X)$-algebra. Let $\{a_1, a_2, \ldots, a_m\}$ be a set of generators of $A_{K(X)}$. For convenience, we may assume that $\{1, X\} \subseteq \{a_1, a_2, \ldots, a_m\}$. $V = K(X)a_1 + K(X)a_2 + \cdots + K(X)a_m$ is a finite dimensional generating subspace of A over $K(X)$. Denote $V' = Ka_1 + Ka_2 + \cdots + Ka_m$ and $K[V']$ = the K-subalgebra of A_K generated by $\{V'\}$. Obviously, $\dim_{K(X)} V \leq \dim_K V'$, for if a subset $\{a_{i'}\}$ of $\{a_i \mid i = 1, 2, \ldots, m\}$ is independent over $K(X)$, then it is certainly independent over K. If a subset $\{a_{i'} a_{j'}\} \subseteq \{a_i a_j \mid 1 \leq i, j \leq m\}$ is $K(X)$-independent, then $\{a_{i'} a_{j'}, Xa_{i'} a_{j'}\}$ is independent over

K. Since $\{a_{i'}a_{j'}, Xa_{i'}a_{j'}\} \subseteq V'^3$, it follows that $2\dim_{K(X)} V^2 \leq \dim_K V'^3$. In the same way, for any n, we have $n \cdot \dim_{K(X)} V^n \leq \dim_K V'^{2n-1}$. So

$$\log_n(n \cdot \dim_{K(X)} V^n) \leq \log_n(\dim_K V'^{2n-1}),$$

$$1 + \log_n \dim_{K(X)} V^n \leq [\log(\dim_K V'^{2n-1})/\log(2n-1)] \cdot [\log(2n-1)/\log n],$$

$$\overline{\lim_{n\to\infty}}\,(1 + \log_n \dim_{K(X)} V^n) \leq \overline{\lim_{n\to\infty}}\,\{[\log(\dim_K V'^{2n-1})/\log(2n-1)] \cdot [\log(2n-1)/\log n]\},$$

Since $\lim_{n\to\infty}[\log(2n-1)/\log n] = 1$,

$$1 + \overline{\lim_{n\to\infty}}\,\log_n(\dim_{K(X)} V^n) \leq \overline{\lim_{n\to\infty}}\,[\log(\dim_K V'^{2n-1})/\log(2n-1)] \leq \overline{\lim_{n\to\infty}}\,\log_n(\dim_K V'^n).$$

Hence $1 + \mathrm{GK}\dim_{K(X)} A \leq \mathrm{GK}\dim_K K[V'] \leq \mathrm{GK}\dim_K A$. ■

In fact, in the above proposition, $\mathrm{GK}\dim_K K[V'] = \mathrm{GK}\dim_K A$, since $A = K(X)[V'] = K[X][V']_{(K[X]/0)^{-1}} = K[V']_{(K[X]/0)^{-1}}$ and $K[X]/0$ consists of regular, central elements of $K[V']$ (Prop. 4.2 [5]).

LEMMA 2. (i) *Let $F \subset K$ be a field extension, A be an algebra over K. Then* $\mathrm{GK}\dim_F A \geq \mathrm{GK}\dim_K A$.

(ii) *If $F \subset K$ is a finite algebraic extension, then* $\mathrm{GK}\dim_F A = \mathrm{GK}\dim_K A$.

The proof is standard. If $F \subset K$ is an infinite algebraic extension, (ii) will not be true, as we will see later, in Example 2.

PROPOSITION 3. *Let $F \subset K$ be a field extension, A be an algebra over K. Then* $\mathrm{GK}\dim_F A \geq \mathrm{GK}\dim_K A + \mathrm{tr}_F(K)$.

PROOF. It suffices to prove the proposition in the case that A is a finitely generated K-algebra and $\mathrm{GK}\dim_K A$ is finite. Let K' be the algebraic closure of F in K, then K is a pure transcendental extension of K'. By Proposition 1 and Lemma 2,

$$\mathrm{GK}\dim_F A \geq \mathrm{GK}\dim_{K'} A \geq \mathrm{GK}\dim_K A + \mathrm{tr}_{K'}(K) = \mathrm{GK}\dim_K A + \mathrm{tr}_F(K). \quad ■$$

As we know, if $F \subset K$ is a field extension, then $\mathrm{GK}\dim_F K = \mathrm{tr}_F(K)$. If A_K is a finitely generated commutative algebra, by the Noetherian Normalization theorem, it is easy to see that $\mathrm{GK}\dim_K A = \mathrm{cl.K.\,dim}\, A = \mathrm{tr}_K(A)$, but $\mathrm{GK}\dim_F A$, in general, is greater than cl.K. dim A. We have

PROPOSITION 4. *Let $F \subset K$ be a field extension, A be an algebra over K. If A is a commutative algebra or a Noetherian P.I. algebra, then* $\mathrm{GK}\dim_F A = \mathrm{GK}\dim_K A + \mathrm{tr}_F(K)$.

PROOF. First, we assume that A is commutative. Without loss of generality, we may assume that A is a finitely generated K-algebra, by definition. Hence A is a Noetherian commutative algebra. (Although there is a direct proof, we reduce it to the case of Noetherian P.I.) So we may assume that A is a Noetherian P.I. algebra. Let N be the nilradical of A.

$$\mathrm{GK}\dim_K A = \mathrm{GK}\dim_K A/N =$$
$$\max\{\mathrm{GK}\dim_K A/P \mid \text{for any minimal prime ideal } P \text{ of } A\} \quad ([6]).$$

Also, we have

$$\mathrm{GK}\dim_F A = \max\{\mathrm{GK}\dim_F A/P \mid \text{for any minimal prime ideal } P \text{ of } A\}.$$

So we may assume further that A is a Noetherian prime P.I. algebra. By Posner's theorem, for any prime P.I. algebra A, A has a simple Artinian quotient algebra Q satisfying the same polynomial identity as A, and

$$Q = A_Z = \{az^{-1} \mid a \in A,\ 0 \neq z \in Z\},$$

where Z is the center of A. And by Kaplansky's theorem, Q is finite dimensional over its center $Z(Q)$. Hence $\mathrm{GK}\dim_K A = \mathrm{GK}\dim_K(Q)$ (Prop. 4.2 [5]) $=$ $\mathrm{GK}\dim_K Z(Q)$ (Prop. 5.5 [5]) $= \mathrm{tr}_K(Z(Q))$. In the same way, $\mathrm{GK}\dim_F A = \mathrm{tr}_F(Z(Q))$. It follows that $\mathrm{GK}\dim_F A = \mathrm{GK}\dim_K A + \mathrm{tr}_F(K)$. ∎

From the above proposition, it seems to make sense to say that Gelfand–Kirillov dimension is a generalization of the "transcendence degree" to several classes of algebras. As we see in the preceding proofs, the behavior of Gelfand–Kirillov dimension under base field extension heavily depends on the relations of generators of A over the base field. If the relations are good enough, for example, group algebra and Weyl algebra, then we have the equality $\mathrm{GK}\dim_F A = \mathrm{GK}\dim_K A + \mathrm{tr}_F(K)$. By the way, if A is an F-algebra, K is a field, then, obviously, $\mathrm{GK}\dim_F A = \mathrm{GK}\dim_K(K \otimes_F A)$.

3. Examples

As with many examples in the theory of Gelfand–Kirillov dimension ([1,4]), our examples are also homomorphic images of free algebra with two generators.

EXAMPLE 1. Let $K(X)$ be the field of rational functions in one indeterminate X over the field K, $K(X)\{Y,Z\}$ be the free algebra with two generators Y and Z over the field $K(X)$. Denote $A = K(X)\{Y,Z\}/(YZ - XZY)$, then $\operatorname{GK}\dim_{K(X)} A = 2$, but $\operatorname{GK}\dim_K A = 4$.

For convenience, let $y = Y + (YZ - XZY)$ and $z = Z + (YZ - XZY)$ in A. Obviously, A can be generated by $\{y,z\}$ as $K(X)$-algebra, and $yz = Xzy$ in A. Notice also $y^i z^j = X^{ij} z^j y^i$ ($\forall i,j \geq 0$) and the monomials $\{z^i y^j \mid i,j \geq 0\}$ are independent over $K(X)$. So, for the generating subspace $V = K(X)y + K(X)z$ of A over $K(X)$, and any integer $n \geq 0$,

$$\dim_{K(X)} V^n = \left| \{z^i y^j \mid i + j = n,\ i,j \geq 0\} \right| = n + 1.$$

$\dim_{K(X)}(K + V + V^2 + \cdots + V^n) = (n+1)(n+2)/2$. It follows that $\operatorname{GK}\dim_{K(X)} A = 2$.

On the other hand, let $V = Ky + Kz$, $K[V]$ be the K-subalgebra of A_K generated by V over K. We claim that $\operatorname{GK}\dim_K K[V] = 4$. Obviously, V^2 is spanned by $\{y^2, Xzy, zy, z^2\}$. It can be verified that for any n, V^n is spanned by $\{X^l z^i y^j \mid i + j = n,\ i,j \geq 0,\ 0 \leq l \leq ij\}$ and this set, in fact, is a basis of V^n over K. So

$$\dim_K V^n = \sum_{i=0}^{n} [i(n-i) + 1].$$

It is easy to know that

$$\dim_K V^n = \binom{n}{3} + \binom{n}{2} + \binom{n}{1} + 1 \qquad \text{for } n \geq 3$$

by Lemma 1.5, [5] ($\binom{n}{k}$ is the binomial coefficients). It follows that

$$d_V(n) = \binom{n}{4} + 2\binom{n}{3} + 2\binom{n}{2} + 2\binom{n}{1} + 1$$

(since $d_V(4) = 30$) for $n \geq 4$. It is a polynomial of n with degree 4. Hence $\operatorname{GK}\dim_K K[V] = 4$.

As we noted before, $\operatorname{GK}\dim_K K(X)[y,z] = \operatorname{GK}\dim_K K[X,y,z]$. Denote $V' = KX + Ky + Kz$, V' is a generating subspace of $K[X,y,z]_K$. It is easy to know that $\dim_K V'^n = \left|\{X^{s+l} z^i y^j \mid i,j,s \geq 0,\ i + j + s = n,\ 0 \leq l \leq ij\}\right|$ and

$$d_{V'}(n) = d_V(n) + 1\cdot n + 2(n-1) + 3(n-3) + \cdots + n\cdot 1$$
$$= d_V(n) + \binom{n}{3} + 2\binom{n}{2} + \binom{n}{1}$$

is also a polynomial of n with degree 4. Hence

$$\text{GK}\dim_K A = \text{GK}\dim_K K(X)[y,z] = \text{GK}\dim_K K[X,y,z] = \overline{\lim_{n\to\infty}}\, d_{V'}(n) = 4. \quad \blacksquare$$

REMARK. In the above example, the main feature which makes $\text{GK}\dim_K A = \text{GK}\dim_{K(X)} A + \text{tr}_K(K(X)) + 1$ is that for any monomial, $z^i y^{n-i}$ comes up $i(n-i)$ times.

EXAMPLE 2. Let $K = F[a_1, a_2, \ldots, a_n, \ldots]$ be an infinite algebraic field extension, a_{i+1} is a square root over $F[a_1, a_2, \ldots, a_i]$ $(i = 1,2,\ldots,n,\ldots)$, a_1 is a square root over the field F. Let $K\{X, Y\}$ be the free algebra with two generators X, Y over K,

$$I = (XYX - a_1X^3, XY^2X - a_2X^4, \ldots, XY^iX - a_iX^{i+2}, \ldots),$$

i.e., the ideal of $K\{X, Y\}$ generated by $\{XY^iX - a_iX^{i+2} \mid i = 1,2,\ldots\}$. Consider $A = K\{X, Y\}/I$. Then A is an algebra over K generated by $x = X + I$ and $y = Y + I$, and $\text{GK}\dim_K A = 3$, $\text{GK}\dim_F A = \infty$.

Notice that $xy^ix = a_ix^{i+2}$ in A and $\{x^iy^j,\ i,j \geq 0\} \cup \{y^ix^jy^k \mid i,j > 0,\ k \geq 0\}$ is a K-basis of A_K. Denote $V = Kx + Ky$. V is a generating subspace of the K-algebra A. Obviously, V^n is spanned by

$$\{x^iy^j \mid i,j \geq 0,\ i + j = n\} \cup \{y^ix^jy^k \mid i,j > 0,\ k \geq 0,\ i + j + k = n\}.$$

Hence

$$\dim_K V^n = n + 1 + n(n-1)/2 = \binom{n}{2} + \binom{n}{1} + 1.$$

It follows that

$$d_V(n) = \dim_K(K + V + \cdots + V^n) = \sum_{i=0}^{n} \dim_K V^i = \binom{n}{3} + 2\binom{n}{2} + 2\binom{n}{1} + 1$$

(since $d_V(3) = 14$). So, $\text{GK}\dim_K A = 3$.

Next, we estimate $\text{GK}\dim_F A$. Denote $V = Fx + Fy$. For any finite dimensional F-subspace V' of A_F, there exist an integer m and some finite extension field K' of F, such that $V' \subseteq K'(F + V + \cdots + V^m)$. Hence $F[V'] \subseteq K'[V]$, $\text{GK}\dim_F F[V'] \leq \text{GK}\dim_F K'[V] = \text{GK}\dim_F F[V]$. By definition, we have

$\mathrm{GK\,dim}_F A = \mathrm{GK\,dim}_F F[V]$. So, to estimate $\mathrm{GK\,dim}_F A$, we first estimate $\dim_F V^n$. Note:

(i) $W_n = \{x^i y^j \mid i,j \geq 0,\ i+j=n\} \cup \{y^i x^j y^k \mid i,j>0,\ k \geq 0,\ i+j+k=n\}$ is an independent subset of V^n, $|W_n| = n+1+n(n-1)/2$.

(ii) Let $W_{n,i}$ ($i \geq 1$) be the subset of W_n consisting of those monomials of W_n whose degree in $x \geq i$. For any monomial w in W_n, if $w \in W_{n,s+2}$, then $a_s w \in V^n$ (for example, if $x^i y^j \in W_n$, $i \geq s+2$, then $a_s x^i y^j = xy^s x x^{i-(s+2)} y^j \in V^n$) and $\bigcup_{s=1}^{n-2} a_s W_{n,s+2}$ is independent over F. In the same way, if $w \in W_{n,s_1+s_2+3}$ ($s_1 < s_2$), then $a_{s_1} a_{s_2} w \in V^n$ and $\bigcup_{s_1+s_2+3 \leq n} a_{s_1} a_{s_2} W_{n,s_1+s_2+3}$ ($s_1 < s_2$) is an F-independent subset of V^n. In general, for any $d > 0$,

$$\bigcup_{s_1+s_2+\cdots+s_d+d+1} (a_{s_1} a_{s_2} \cdots a_{s_d} W_{n,s_1+s_2+\cdots+s_d+d+1}) \quad (s_1 < s_2 < \cdots < s_d)$$

is an F-independent subset of V^n.

In particular, for any $d > 3$ and sufficient large n,

$$S = \{a_{s_1} a_{s_2} \cdots a_{s_d} x^{n+d+1} \mid s_1 + s_2 + \cdots + s_d + d + 1 = n + d + 1,$$
$$1 \leq s_1 < s_2 < \cdots < s_d\}$$

is an F-independent subset of V^{n+d+1}. Hence

$$\dim_F V^{n+d+1} > |S| > 1/d!\left[\binom{n-1}{d-1} - \binom{n-3}{d-3} - \binom{n-5}{d-3} - \binom{n-7}{d-3} - \cdots\right]$$
$$> 1/d!\left\{\binom{n-1}{d-1} - \left[\binom{n-3}{d-3} + \binom{n-4}{d-3} + \cdots + \binom{d-3}{d-3}\right]\right\}$$
$$= 1/d!\binom{n-1}{d-1} - \lambda \cdot *, \qquad 0 < \lambda \in \mathbf{Q}$$

and $*$ is a polynomial of n with degree $d-2$. It follows that

$$d_V(n+d+1) = \dim_F(F + V + \cdots + V^{n+d+1}) > \sum_{i=0}^{n} \dim_F V^{i+d+1}$$
$$\geq \text{some polynomial of } n \text{ with degree } d.$$

Hence

$$\mathrm{GK\,dim}_F A = \varlimsup_{n\to\infty} d_V(n) \geq \varlimsup_{n\to\infty} d_V(n+d+1) \geq d.$$

So, $\mathrm{GK\,dim}_F A = \infty$. ∎

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